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Quantum treatment of the time-dependent coupled oscillators

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Abstract. The Hamiltonian for a system of two coupled oscillators with time-dependent coupling parameter and the phase pump is considered. The wavefunction in Schrödinger picture and the Green function are calculated. The squeezing phenomena as well as the Glauber second-order correlation function is discussed. Statistical investigation for the quasi-probability distribution function (P-representation, W-Wigner and Q-functions) are given.

1. Introduction

The problem of a frequency converter and a parametric amplifier, where three electromagnetic fields are coupled [1–8] represents one of the relevant problem to the field of quantum optics. This problem represents an important nonlinear parametric interaction, which has played a significant role in several physical phenomena of interest, such as stimulated and spontaneous emissions of radiation, coherent Raman and Brillouin scattering. In Brillouin scattering one finds that an intense monochromatic laser source induces parametric coupling between the two scattered electromagnetic fields and the acoustical phonons in the scattering medium. In Raman scattering a similar coupling occurs between the scattered Stokes and anti-Stokes waves and the optical phonons of a Raman active medium [3, 5, 8]. The most familiar Hamiltonian representing such a system is given by

$$\frac{H}{\hbar} = \omega_1 a^\dagger a + \omega_2 b^\dagger b + \omega_3 c^\dagger c + k(ab^\dagger c^\dagger + a^\dagger bc)\,. \tag{1.1}$$

Equation (1.1) can be looked upon as a frequency converter model in which the idler photon is the sum frequency of a laser photon and a signal photon, provided we identify $a(t), b(t)$ and $c(t)$ as the annihilation operators of the idler, signal and laser modes, respectively. It is interesting to note that if one takes $J_+ = ba^\dagger$, $\;J_- = b^\dagger a$, where J is the collective angular momentum operator, the Hamiltonian (1.1) can be compared with the Hamiltonian which represents coherent emission from a system of N two-level atoms interacting with a single mode of the radiation field. The Hamiltonian representing such a system takes the form

$$\frac{H}{\hbar} = \omega c^\dagger c + \frac{1}{2}\omega_0 J_z + k(cJ_+ + c^\dagger J_-) \tag{1.2}$$

which is known as the Tavis–Cummings model [9].

To discuss the dynamical behaviour of the above system (equations (1.1) or (1.2)) one needs to solve either the Heisenberg or the Schrödinger equations. However, as a result of the existence of the nonlinearity terms in the interaction part of the Hamiltonians, the

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solution cannot be an easy task to find. Nevertheless, under certain approximations the above equations can be linearized. For example, equation (1.1) can be linearized if one takes the laser mode to be in a coherent state, and of sufficient intensity that we may neglect the reaction of the nonlinear coupling back on the state of this mode. In this parametric approximation, the field operators for the laser mode may be replaced by their expectation values, such that

$$c(t) \to \bar{c}(t)\exp(-\mathrm{i}\phi(t)) \tag{1.3}$$

and hence (1.1) becomes

$$\frac{H}{\hbar} = \omega_1 a^\dagger a + \omega_2 b^\dagger b + \lambda(t)\left(ab^\dagger \mathrm{e}^{\mathrm{i}\phi(t)} + a^\dagger b\mathrm{e}^{-\mathrm{i}\phi(t)}\right) . \tag{1.4}$$

On the other hand, if one uses the Holstein–Primakoff transformation [10]

$$J_- = \sqrt{J - \hat{n}_b}\; b\mathrm{e}^{-\mathrm{i}\phi(t)} \tag{1.5a}$$

$$J_+ = b^\dagger\sqrt{J - \hat{n}_b}\; \mathrm{e}^{\mathrm{i}\phi(t)} \tag{1.5b}$$

and

$$\hat{n}_b = b^\dagger b = \tfrac{1}{2}(J + J_z) . \tag{1.5c}$$

The Hamiltonian (1.2) can also be linearized if we approximate $\hat{n}_b$ with its c-number time-dependent function $n_b(t)$, this approximation is applicable if $\hat{n}_b - n_b$ can be treated as small perturbations. In this case equation (1.2) takes a form similar to that given by equation (1.4); for more details see [11–14]. It is noteworthy that the authors in [15, 16] considered the same Hamiltonian (1.4) in order to discuss the finite coherence time of a continuous laser pump. These authors have allowed the pump amplitude and phase to be arbitrary time-dependent functions rather than constants. Since the time-dependent Hamiltonian model does not acquire any dispersion processes, the commutation relation for the operators a and b with their conjugate are therefore preserved under the present Hamiltonian and satisfy

$$[a, a^\dagger] = 1 = [b, b^\dagger] \tag{1.6a}$$

$$[a, b] = 0 = [a^\dagger, b^\dagger]. \tag{1.6b}$$

In previous literature the problem of two coupled oscillators have been extensively considered by many authors, with the emphasis on discussion of the statistical properties, such as the photon number, as well as the P-representation [17–19]. Most workers have used the solution of the Heisenberg equations of motion, taking the coupling parameter $\lambda(t)$ to be constant, and the phase pump $\phi(t) = \omega t$. In the present paper we shall handle the same problem, but without any restriction on either the coupling parameter $\lambda(t)$ or the phase $\phi(t)$, except for the integrability condition, which we shall impose in order to find the general solution of the problem. The problem will be considered in two different parts. The first is to find the solution in the Schrödinger picture as well as the accurate definition of the Dirac operator where the Hamiltonian (1.4) can be diagonalized; we shall also consider the calculation of the Green function: this will be done in sections 2 and 3, respectively. In the second part we shall fill the gap left in [14] by devoting section 4 to discussing the squeezing phenomenon, using the correlated squeezed operator model as well as even coherent states. Also we shall extend our discussion to include the second-order correlation function $g^{(2)}(t)$; this will be given in section 5. In section 6 we shall consider the quasi-probability distribution functions; our conclusions follow in section 7.

2. The Schrödinger wavefunction

In this section we shall pay attention to find the exact solution of the wavefunction in the Schrödinger picture (pseudo-stationary state), for the Hamiltonian (1.4). To do so, we shall introduce two pairs of Dirac operators, namely

$$a = (2\hbar\omega_1)^{-\frac{1}{2}}(\omega_1 q_1 + \mathrm{i}p_1) \tag{2.1a}$$

$$b = (2\hbar\omega_2)^{-\frac{1}{2}}(\omega_2 q_2 + \mathrm{i}p_2)\,. \tag{2.1b}$$

From equations (1.4) and (2.1) we have

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) + \frac{\lambda(t)}{\sqrt{\omega_1\omega_2}}[(\omega_1\omega_2 q_1 q_2 + p_1 p_2)\cos\phi(t)$$
$$+(\omega_1 q_1 p_2 - \omega_2 p_1 q_2)\sin\phi(t)]\,. \tag{2.2}$$

The Schrödinger equation for the time-dependent Hamiltonian H is given by

$$H\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}\,. \tag{2.3}$$

Therefore, inserting equation (2.2) in (2.3) yields

$$\frac{\partial^2\psi}{\partial q_1^2} + \frac{\partial^2\psi}{\partial q_2^2} - \frac{1}{\hbar^2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2)\psi - 2\frac{\lambda(t)}{\hbar}\left[\sqrt{\omega_1\omega_2}q_1 q_2\psi - \frac{\hbar^2}{\sqrt{\omega_1\omega_2}}\frac{\partial^2\psi}{\partial q_1\partial q_2}\right]\cos\phi(t)$$
$$+\frac{2\mathrm{i}}{\hbar}\lambda(t)\left[\left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}} q_1\frac{\partial\psi}{\partial q_2} - \left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}} q_2\frac{\partial\psi}{\partial q_1}\right]\sin\phi(t) = -\frac{2\mathrm{i}}{\hbar}\frac{\partial\psi}{\partial t}\,. \tag{2.4}$$

In order to solve the above equation we shall introduce the following transformations:

$$\sqrt{\omega_1}q_1 = Q_1\cos\gamma_+(t) + P_1\sin\gamma_+(t) \tag{2.5a}$$

$$p_1/\sqrt{\omega_1} = P_1\cos\gamma_+(t) - Q_1\sin\gamma_+(t) \tag{2.5b}$$

$$\sqrt{\omega_2}q_2 = Q_2\cos\gamma_-(t) + P_2\sin\gamma_-(t) \tag{2.5c}$$

$$p_2/\sqrt{\omega_2} = P_2\cos\gamma_-(t) - Q_2\sin\gamma_-(t) \tag{2.5d}$$

where

$$\gamma_\pm(t) = \tfrac{1}{2}[(\omega_1 + \omega_2)t \pm \phi(t)]\,. \tag{2.6}$$

From equations (2.4) and (2.5) the wavefunction will take the form

$$\Delta(t)\left[\left(\frac{\partial^2\bar\psi}{\partial Q_1^2} - \frac{\partial^2\bar\psi}{\partial Q_2^2}\right) - \frac{1}{\hbar^2}(Q_1^2 - Q_2^2)\bar\psi\right] + 2\frac{\lambda(t)}{\hbar^2}Q_1 Q_2\bar\psi - 2\lambda(t)\frac{\partial^2\bar\psi}{\partial Q_1\partial Q_2} = \frac{2\mathrm{i}}{\hbar}\frac{\partial\bar\psi}{\partial t} \tag{2.7}$$

where

$$\Delta(t) = \tfrac{1}{2}(\dot\phi(t) + \omega_2 - \omega_1)\,. \tag{2.8}$$

Let us now introduce the integrability condition, which should be imposed in order to find the general solution of (2.7), that is

$$2\theta\lambda(t) = (\omega_1 - \omega_2 - \dot\phi(t)) \tag{2.9}$$

where θ is an arbitrary non-zero constant. Here we may refer to [14], where this condition has been derived, the reason for imposing it (where the solution of differential equations resulting from the Hamiltonian similar to that given by (1.4) is obtained) having been well established. Although the integrability condition allows us to give an exact solution in a compact form; however, it sometimes leads to the complicated situation of studying some phenomena such as periodic coupling. This can be seen if one tries to find the photon numbers in the transformed Tavis–Cummings model where $\lambda(t) = k\sqrt{J - n_b(t)}$.

For example, the authors of [13] considered this case by ignoring the integrability condition, as a result of the complication, and discussed the problem at exact resonance in the absence of the phase pump $\phi(t)$. The left-hand side of (2.9) is proportional to the electric field of the laser, while the right-hand side is the energy conservation plus the laser phase derivative; this in fact represents a sufficient condition for continuous periodic energy exchange between the modes where the periods are nonlinear functions of time. Furthermore, if one uses the transformation

$$x = \frac{1}{\sqrt{\hbar}}(Q_1 \cos\delta + Q_2 \sin\delta) \tag{2.10a}$$

$$y = \frac{1}{\sqrt{\hbar}}(Q_2 \cos\delta - Q_1 \sin\delta) \tag{2.10b}$$

where δ is the angle of rotation defined by $\delta = \frac{1}{2}\cot^{-1}\theta$, then the wavefunction $\bar{\psi}(Q_1, Q_2, t) \to \eta(x, y, t)$ and

$$\frac{\partial\bar{\psi}}{\partial Q_1} = \frac{1}{\sqrt{\hbar}}\left(\cos\delta\frac{\partial\eta}{\partial x} - \sin\delta\frac{\partial\eta}{\partial y}\right) \tag{2.11a}$$

$$\frac{\partial\bar{\psi}}{\partial Q_2} = \frac{1}{\sqrt{\hbar}}\left(\cos\delta\frac{\partial\eta}{\partial y} + \sin\delta\frac{\partial\eta}{\partial x}\right). \tag{2.11b}$$

By substituting equations (2.10) and (2.11) in (2.7) one obtains

$$\frac{\partial^2\eta}{\partial x^2} - \frac{\partial^2\eta}{\partial y^2} - (x^2 - y^2)\eta = -\frac{2\mathrm{i}}{\lambda(t)}\sin 2\delta\frac{\partial\eta}{\partial t}. \tag{2.12}$$

Once we obtain the solution of (2.12), which is easy to solve, we are then in a position to find the general solution of (2.7); thus

$$\begin{aligned}\bar{\psi}_{mn}(Q_1, Q_2, t) = {} & (\pi\hbar)^{-\frac{1}{2}}2^{-\frac{1}{2}(n+m)}(n!m!)^{-\frac{1}{2}}\exp\left[-\frac{1}{2\hbar}(Q_1^2 + Q_2^2)\right] \\ & \times H_n\left[\frac{1}{\sqrt{\hbar}}(Q_1\cos\delta + Q_2\sin\delta)\right] H_m\left[\frac{1}{\sqrt{\hbar}}(Q_2\cos\delta - Q_1\sin\delta)\right] \\ & \times \exp[-\mathrm{i}(n-m)I(t)]\end{aligned} \tag{2.13}$$

with $I(t) = \sqrt{\theta^2 + 1}\int_0^t \lambda(t')\,\mathrm{d}t'$, and $H(\cdots)$ denotes the Hermite polynomial.

On reverting to physical coordinates, we have to calculate the integral

$$\psi_{nm}(q_1, q_2, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\psi}_{nm}(Q_1, Q_2, t)\tilde{K}(q_1, q_2, Q_1, Q_2, t)\,\mathrm{d}Q_1\,\mathrm{d}Q_2 \tag{2.14}$$

where $\tilde{K}$ represents the kernel which can be calculated if one uses (2.5). In this case we find

$$\begin{aligned}\tilde{K}(q_1, q_2, Q_1, Q_2, t) = {} & (\omega_1\omega_2)^{\frac{1}{4}} \Big/ \left[2\hbar\pi\sqrt{\sin\gamma_+(t)\sin\gamma_-(t)}\right] \\ & \times \exp\left(\frac{\mathrm{i}}{2\hbar}[(Q_1^2 + \omega_1 q_1^2)\cot\gamma_+(t) - 2\sqrt{\omega_1}q_1 Q_1\operatorname{cosec}\gamma_+(t)]\right)\end{aligned}$$

$$\times \exp\left(\frac{\mathrm{i}}{2\hbar}[(Q_2^2+\omega_2 q_2^2)\cot\gamma_-(t) - 2\sqrt{\omega_2}q_2 Q_2 \operatorname{cosec}\gamma_-(t)]\right). \tag{2.15}$$

If we now insert equations (2.13) and (2.15) into (2.14), after performing the integal we obtain

$$\begin{aligned}
\psi_{mn}(q_1,q_2,t) = {} & \frac{(\omega_1\omega_2)^{\frac{1}{4}}}{\sqrt{\pi\hbar}} 2^{-\frac{1}{2}(n+m)}(n!m!)^{\frac{1}{2}}(\cos\phi(t)+\mathrm{i}\sin\phi(t)\cos 2\delta)^{\frac{1}{2}m} \\
& \times(\cos\phi(t)-\mathrm{i}\sin\phi(t)\cos 2\delta)^{\frac{1}{2}n}\exp\left[-\frac{1}{2\hbar}(\omega_1 q_1^2+\omega_2^2 q_2^2)\right] \\
& \exp\left(-\mathrm{i}\left[\mu_-(t)\left(m+\frac{1}{2}\right)+\mu_+(t)\left(n+\frac{1}{2}\right)\right]\right) \\
& \times\sum_{l=0}^{m}[l!(n-l)!(m-l)!]^{-1}\left[\frac{-2\mathrm{i}\sin 2\delta\sin\phi(t)}{\sqrt{\cos^2\phi(t)+\sin^2\phi(t)\cos^2 2\delta}}\right]^l \\
& \times H_{(n-l)}\left[\frac{1}{\sqrt{\hbar}}\left(\frac{\sqrt{\omega_1}q_1\cos\delta\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi(t)}+\sqrt{\omega_2}q_2\sin\delta\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi(t)}}{(\cos\phi(t)-\mathrm{i}\sin\phi(t)\cos 2\delta)^{\frac{1}{2}}}\right)\right] \\
& \times H_{(m-l)}\left[\frac{1}{\sqrt{\hbar}}\left(\frac{\sqrt{\omega_2}q_2\cos\delta\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi(t)}-\sqrt{\omega_1}q_1\sin\delta\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi(t)}}{(\cos\phi(t)+\mathrm{i}\sin\phi(t)\cos 2\delta)^{\frac{1}{2}}}\right)\right]
\end{aligned} \tag{2.16a}$$

where

$$\mu_\pm(t) = \tfrac{1}{2}(\omega_1+\omega_2)t \pm I(t). \tag{2.16b}$$

The connection between the wavefunction in the Schrödinger representation (psuedo-stationary state), and the wavefunction in the quasi-coherent state can be found from the relation

$$\psi_{\alpha\beta}(q_1,q_2,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{e}^{-\frac{1}{2}(|\alpha^2|+|\beta^2|)}\frac{\alpha^n\beta^m}{\sqrt{n!m!}}\psi_{nm}(q_1,q_2,t). \tag{2.17}$$

From equations (2.16*a*) and (2.17), after some calculation, we obtain the following expression:

$$\begin{aligned}
\psi_{\alpha\beta}(q_1,q_2,t) = {} & \left(\frac{\omega_1\omega_2}{(\hbar\pi)^2}\right)^{\frac{1}{4}}\exp\left[-\frac{1}{2\hbar}(\omega_1 q_1^2+\omega_2 q_2^2)\right] \\
& \times\exp\left[\sqrt{\frac{2}{\hbar}}\alpha(t)\left(\sqrt{\omega_1}q_1\cos\delta\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi(t)}+\sqrt{\omega_2}q_2\sin\delta\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi(t)}\right)\right] \\
& \times\exp\left[\sqrt{\frac{2}{\hbar}}\beta(t)\left(\sqrt{\omega_2}q_2\cos\delta\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi(t)}-\sqrt{\omega_1}q_1\sin\delta\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi(t)}\right)\right] \\
& \times\exp[-\mathrm{i}\alpha(t)\beta(t)\sin 2\delta\sin\phi(t)] \\
& \times\exp\left[-\tfrac{1}{2}\alpha^2(t)(\cos\phi(t)-\mathrm{i}\sin\phi(t)\cos 2\delta)\right] \\
& \times\exp\left[-\tfrac{1}{2}\beta^2(t)(\cos\phi(t)+\mathrm{i}\sin\phi(t)\cos 2\delta)\right]\exp\left[-\tfrac{1}{2}(|\alpha^2|+|\beta^2|)\right]
\end{aligned} \tag{2.18}$$

where

$$\alpha(t) = \alpha(0)\exp(-\mathrm{i}\mu_+(t)) \tag{2.19a}$$

$$\beta(t) = \beta(0)\exp(-\mathrm{i}\mu_-(t))\,. \tag{2.19b}$$

In section 3, we shall turn our attention to constructing the 'best' Dirac operators, which can be used to diagonalize the Hamiltonian (2.2).

3. The diagonalized Hamiltonian and the Green function

In order to diagonalize the Hamiltonian (2.2) we have to find the accurate definition for the Dirac operators, this can be done by using (2.18) from the previous section.

By differentiating equation (2.18) partially with respect to q_1 and q_2, and using (2.17), we may construct two pairs of the Dirac operators in the form

$$A(t) = (2\hbar)^{-\frac{1}{2}}\left[\frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi(t)}}{\sqrt{\omega_1}}(\omega_1 q_1 + \mathrm{i}p_1)\cos\delta + \frac{\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi(t)}}{\sqrt{\omega_2}}(\omega_2 q_2 + \mathrm{i}p_2)\sin\delta\right] \tag{3.1a}$$

$$B(t) = (2\hbar)^{-\frac{1}{2}}\left[\frac{\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi(t)}}{\sqrt{\omega_2}}(\omega_2 q_2 + \mathrm{i}p_2)\cos\delta - \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi(t)}}{\sqrt{\omega_1}}(\omega_1 q_1 + \mathrm{i}p_1)\sin\delta\right] \tag{3.1b}$$

which satisfy the commutation relations

$$[A, B] = 0 = [A, B^\dagger] \tag{3.1c}$$

$$[A, A^\dagger] = 1 = [B, B^\dagger] \tag{3.1d}$$

for all values of time t.

Substituting equation (3.1) in (1.4), with aid of (2.1) we have

$$\begin{aligned}\frac{H}{\hbar} = {} & (\omega_1\cos^2\delta + \omega_2\sin^2\delta + \lambda(t)\sin 2\delta)A^\dagger A \\ & + (\omega_2\cos^2\delta + \omega_1\sin^2\delta - \lambda(t)\sin 2\delta)B^\dagger B \\ & + \left(\frac{\omega_2-\omega_1}{2}\sin 2\delta + \lambda(t)\cos 2\delta\right)(A^\dagger B + B^\dagger A) + \frac{\partial F_2}{\partial t}\end{aligned} \tag{3.2}$$

where F_2 is the generating function given by

$$2F_2 = \phi(t)[\sin 2\delta(A^\dagger B + B^\dagger A) + \cos 2\delta(B^\dagger B - A^\dagger A)]\,. \tag{3.3}$$

From equations (3.2) and (3.3), we get

$$\begin{aligned}\frac{H}{\hbar} = {} & (\omega_1\cos^2\delta + \omega_2\sin^2\delta + \lambda(t)\sin 2\delta - \frac{\dot\phi(t)}{2}\cos 2\delta)A^\dagger A \\ & + (\omega_2\cos^2\delta + \omega_1\sin^2\delta - \lambda(t)\sin 2\delta + \frac{\dot\phi(t)}{2}\cos 2\delta)B^\dagger B \\ & + \left[\frac{1}{2}(\omega_2 - \omega_1 + \dot\phi(t))\sin 2\delta + \lambda(t)\cos 2\delta\right](A^\dagger B + B^\dagger A)\,.\end{aligned} \tag{3.4}$$

By using the integrability condition given by (2.9), we may obtain the diagonalized Hamiltonian in the following form:

$$\frac{H}{\hbar} = \dot\mu_+(t)A^\dagger A + \dot\mu_-(t)B^\dagger B \tag{3.5}$$

where $\mu_\pm(t)$ is defined by (2.16*b*), and the dot indicates the first derivative of the function $\mu_\pm(t)$ with respect to time.

Now we shall extend our progress to include calculation of the Green function, which can be obtained if one uses the formula

$$G(q_1, q_2, \bar{q}_1, \bar{q}_2, t) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \psi_{\alpha\beta}(q_1, q_2, t)\psi^*_{\alpha\beta}(\bar{q}_1, \bar{q}_2, 0)\, \mathrm{d}^2\alpha\, \mathrm{d}^2\beta \tag{3.6}$$

where $\psi^*_{\alpha\beta}(\bar{q}_1, \bar{q}_2, 0)$ is the complex conjugate of (2.18) at the time $t = 0$. However, we may obtain the Green function by using the solution in the Heisenberg picture; for the Hamiltonian (2.2) this solution takes the form

$$\begin{aligned} q_1(t) = q_1(0)&[\cos I(t)\cos\gamma_+(t) - \sin I(t)\sin\gamma_+(t)\cos 2\delta] \\ &+ \frac{p_1(0)}{\omega_1}[\cos\gamma_+(t)\cos I(t)\cos 2\delta + \cos I(t)\sin\gamma_+(t)] \\ &+ \frac{p_2(0)}{\sqrt{\omega_1\omega_2}}\sin 2\delta\sin I(t)\cos\gamma_+(t) - \sqrt{\frac{\omega_2}{\omega_1}}q_2(0)\sin 2\delta\sin I(t)\sin\gamma_+(t) \end{aligned} \tag{3.7a}$$

$$\begin{aligned} p_1(t) = p_1(0)&[\cos I(t)\cos\gamma_+(t) - \sin I(t)\sin\gamma_+(t)\cos 2\delta] \\ &- \omega_1 q_1(0)[\cos\gamma_+(t)\cos 2\delta\sin I(t) + \cos I(t)\sin\gamma_+(t)] \\ &- \sqrt{\omega_1\omega_2}q_2(0)\sin 2\delta\sin I(t)\cos\gamma_+(t) - \sqrt{\frac{\omega_1}{\omega_2}}p_2(0)\sin 2\delta\sin I(t)\sin\gamma_+(t) \end{aligned} \tag{3.7b}$$

$$\begin{aligned} q_2(t) = q_2(0)&[\cos I(t)\cos\gamma_-(t) + \cos 2\delta\sin I(t)\sin\gamma_-(t)] \\ &- \frac{p_2(0)}{\omega_2}[\cos 2\delta\cos\gamma_-(t)\sin I(t) - \cos I(t)\sin\gamma_-(t)] \\ &+ \frac{p_1(0)}{\sqrt{\omega_1\omega_2}}\sin 2\delta\sin I(t)\cos\gamma_-(t) - \sqrt{\frac{\omega_1}{\omega_2}}q_1(0)\sin 2\delta\sin I(t)\sin\gamma_-(t) \end{aligned} \tag{3.7c}$$

$$\begin{aligned} p_2(t) = p_2(0)&[\cos I(t)\cos\gamma_-(t) + \cos 2\delta\sin I(t)\sin\gamma_-(t)] \\ &+ \omega_2 q_2(0)[\cos 2\delta\cos\gamma_-(t)\sin I(t) - \sin\gamma_-(t)\cos I(t)] \\ &- \sqrt{\omega_1\omega_2}q_1(0)\sin 2\delta\sin I(t)\cos\gamma_-(t) - \sqrt{\frac{\omega_2}{\omega_1}}p_1(0)\sin 2\delta\sin I(t)\sin\gamma_-(t) \end{aligned} \tag{3.7d}$$

where $\gamma_\pm(t)$ is given by (2.6).

It is easy to show that $[q_i, p_i] = \mathrm{i}\hbar\delta_{ij}$, with $\delta_{ij} = 1$ if $i = j$ and zero otherwise. The calculation of the Green function gives the following result:

$$\begin{aligned} G(q_1, q_2, \bar{q}_1, \bar{q}_2, t) = \left[\frac{\omega_1\omega_2}{l}\right]^{\frac{1}{2}} \exp\bigg(&\frac{-\mathrm{i}\omega_1}{2\hbar l}[\sin(\omega_1+\omega_2)t + \sin^2\delta\sin(2I(t) - \phi(t)) \\ &- \cos^2\delta\sin(2I(t) + \phi(t))]q_1^2\bigg) \end{aligned}$$

$$
\times \exp\left(\frac{-\mathrm{i}\omega_2}{2\hbar l}[\sin(\omega_1+\omega_2)t+\cos^2\delta\sin(2I(t)+\phi(t))\right.
$$

$$
\left.-\sin^2\delta\sin(2I(t)-\phi(t))]q_2^2\right)
$$

$$
\times \exp\left[\left(\frac{\mathrm{i}\sqrt{\omega_1\omega_2}}{2\hbar l}\sin 2\delta\sin 2I(t)\right)q_1q_2\right]
$$

$$
\times \exp\left(\frac{-\mathrm{i}\omega_1}{\hbar l}\left[\sin\gamma_-(t)\cos\gamma_+(t)+\sin^2 I(t)\sin\phi(t)\cos^2 2\delta\right.\right.
$$

$$
\left.\left.-\frac{1}{2}\sin 2I(t)\cos\phi(t)\cos 2\delta\right]\bar{q}_1^2\right)
$$

$$
\times \exp\left(\frac{-\mathrm{i}\omega_2}{\hbar l}\left[\sin\gamma_+(t)\cos\gamma_-(t)-\sin^2 I(t)\sin\phi(t)\cos^2 2\delta\right.\right.
$$

$$
\left.\left.+\frac{1}{2}\sin 2I(t)\cos\phi(t)\cos 2\delta\right]\bar{q}_2^2\right)
$$

$$
\times \exp\left(\frac{2\mathrm{i}\sqrt{\omega_1\omega_2}}{\hbar l}\left[\sin 2\delta\sin I(t)\left(\cos^2\delta\cos(I(t)+\phi(t))\right.\right.\right.
$$

$$
\left.\left.\left.+\sin^2\delta\cos(I(t)-\phi(t))\right)\bar{q}_1\bar{q}_2\right]\right)
$$

$$
\times \exp\left(\frac{2\mathrm{i}\omega_1}{\hbar l}\left[\sin^2\delta\sin(I(t)+\gamma_-(t))-\cos^2\delta\sin(I(t)-\gamma_-(t))\right]q_1\bar{q}_1\right)
$$

$$
\times \exp\left(\frac{2\mathrm{i}\omega_2}{\hbar l}\left[\cos^2\delta\sin(I(t)+\gamma_+(t))-\sin^2\delta\sin(I(t)-\gamma_+(t))\right]q_2\bar{q}_2\right)
$$

$$
\times \exp\left[\left(\frac{-2\mathrm{i}\sqrt{\omega_1\omega_2}}{\hbar l}\sin 2\delta\sin I(t)\cos\gamma_+(t)\right)\bar{q}_1q_2\right]
$$

$$
\times \exp\left[\left(\frac{-2\mathrm{i}\sqrt{\omega_1\omega_2}}{\hbar l}\sin 2\delta\sin I(t)\cos\gamma_-(t)\right)q_1\bar{q}_2\right] \tag{3.8}
$$

where l is given by

$$
l=[\cos(\omega_1+\omega_2)t-\cos^2\delta\cos(2I(t)+\phi(t))-\sin^2\delta\cos(2I(t)-\phi(t))]\,. \tag{3.9}
$$

As a special case if we take $\lambda(t)$ to be constant, and $\phi(t)=\omega t$, we obtain equation (3.7) of [20].

4. Squeezing

Now we shall employ the solution in the Heisenberg picture to discuss the squeezing phenomena. This phenomenon is characterized by fluctuations in the quadrature of the field being less than the vacuum fluctuation [21, 22]. To examine the squeezing one needs to calculate the quadrature variances in each mode; this can be done by using (3.7). To do so, let us rewrite equation (3.7) in the Dirac representation; thus

$$a(t) = \mathrm{e}^{-\mathrm{i}\gamma_+(t)}[a(0)(\cos I(t) - \mathrm{i}\cos 2\delta \sin I(t)) - \mathrm{i}\sin 2\delta \sin I(t) b(0)] \tag{4.1a}$$

$$b(t) = \mathrm{e}^{-\mathrm{i}\gamma_-(t)}[b(0)(\cos I(t) + \mathrm{i}\sin I(t)\cos 2\delta) - \mathrm{i}\sin 2\delta \sin I(t) a(0)]\,. \tag{4.1b}$$

If the system is considered to be initially in the coherent or in the vacuum states, we shall find there is no squeezing, this of course is due to the nature of the system. Therefore we shall consider the following cases.

4.1. A two-mode squeezed coherent state

A two-mode squeezed coherent state is defined [23] as

$$|\alpha, \beta, r\rangle \equiv \hat{D}(\alpha)\hat{D}(\beta)\hat{S}(r)|0, 0\rangle \tag{4.2}$$

where the squeeze operator $\hat{S}(r)$ is given by

$$\hat{S}(r) = \exp[r(a^\dagger b^\dagger - ab)] \tag{4.3}$$

and the Glauber displacement operators $\hat{D}(\alpha)$ and $\hat{D}(\beta)$ [24] by

$$\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{4.4a}$$

$$\hat{D}(\beta) = \exp(\beta b^\dagger - \beta^* b)\,. \tag{4.4b}$$

The squeezing operators provide a Bogoliubov transformation of the annihilation operators as

$$\hat{S}(r)^{-1} a \hat{S}(r) = a \cosh r + b^\dagger \sinh r \tag{4.5a}$$

$$\hat{S}(r)^{-1} b \hat{S}(r) = b \cosh r + a^\dagger \sinh r \tag{4.5b}$$

while the Glauber displacement operators $\hat{D}(\alpha)$ and $\hat{D}(\beta)$ produce the operator transformations

$$\hat{D}^{-1}(\alpha) a \hat{D}(\alpha) = a + \alpha \tag{4.6a}$$

$$\hat{D}^{-1}(\beta) b \hat{D}(\beta) = b + \beta\,. \tag{4.6b}$$

Note that for convenience the squeeze parameter r has been taken to be real.
Now let us define two quadrature operators

$$X_1 = \tfrac{1}{2}(a + a^\dagger) \tag{4.7a}$$

$$Y_1 = \frac{1}{2\mathrm{i}}(a - a^\dagger) \tag{4.7b}$$

$$X_2 = \tfrac{1}{2}(b + b^\dagger) \tag{4.7c}$$

$$Y_2 = \frac{1}{2\mathrm{i}}(b - b^\dagger)\,. \tag{4.7d}$$

Then, the quadrature variances can be written as

$$\overline{\Delta X_1}^2 = \tfrac{1}{4}\left[\cosh 2r - (\sin 2\delta \sin 2I(t) \sin 2\gamma_+(t) + \sin 4\delta \sin^2 I(t) \cos 2\gamma_+(t)) \sinh 2r\right] \tag{4.8a}$$

$$\overline{\Delta Y_1}^2 = \tfrac{1}{4}\left[\cosh 2r + (\sin 2\delta \sin 2I(t) \sin 2\gamma_+(t) + \sin 4\delta \sin^2 I(t) \cos 2\gamma_+(t)) \sinh 2r\right]\,. \tag{4.8b}$$

For the second mode we have

$$\overline{\Delta X_2}^2 = \tfrac{1}{4}\left[\cosh 2r - (\sin 2\delta \sin 2I(t)\sin 2\gamma_-(t) - \sin 4\delta \sin^2 I(t)\cos 2\gamma_-(t))\sinh 2r\right] \tag{4.9a}$$

$$\overline{\Delta Y_2}^2 = \tfrac{1}{4}\left[\cosh 2r + (\sin 2\delta \sin 2I(t)\sin 2\gamma_-(t) - \sin 4\delta \sin^2 I(t)\cos 2\gamma_-(t))\sinh 2r\right]\ . \tag{4.9b}$$

From equation (4.8) we may conclude that, the squeezing is occuring in the first quadrature provided the term multiplied by $\sinh 2r$ is positive; however, as a result of the oscillating terms, exchange between the two quadratures is excepted. The same conclusion would apply to the second mode, see equation (4.9). As a special case, if we take $\theta \to 0$, so that $\delta = \frac{\pi}{4}$ and λ becomes constant, then equation (4.8) gives

$$\overline{\Delta X_1}^2 = \tfrac{1}{4}[\cosh 2r - \sin 2\lambda t \sin 2\omega_1 t \sinh 2r] \tag{4.10a}$$

$$\overline{\Delta Y_1}^2 = \tfrac{1}{4}[\cosh 2r + \sin 2\lambda t \sin 2\omega_1 t \sinh 2r]\ . \tag{4.10b}$$

The above equations can be compared with equations (24*a*), (24*b*) of [25]. On the other hand, if we take the coupling parameter λ to be constant, while $\phi(t) = -(\omega_1 + \omega_2)t$. The quadrature variances for the first mode 'a' become

$$\overline{\Delta X_1}^2 = \frac{1}{4}\left[\cosh 2r - \frac{2\omega_1\lambda}{(\omega_1^2+\lambda^2)}\sin^2\sqrt{\omega_1^2+\lambda^2}t \sinh 2r\right] \tag{4.11a}$$

$$\overline{\Delta Y_1}^2 = \frac{1}{4}\left[\cosh 2r + \frac{2\omega_1\lambda}{(\omega_1^2+\lambda^2)}\sin^2\sqrt{\omega_1^2+\lambda^2}t \sinh 2r\right] \tag{4.11b}$$

which shows that for $r > 0$ we have no exchange between the quadrature variances and therefore the squeezing will remain in the first quadrature. The same argument can be applied for the second mode 'b' provided that $\phi(t) = (\omega_1 + \omega_2)t$ and λ is constant. In this case the result will be the same as the result given by equations (4.11*a*), (4.11*b*), but with ω_2 replaced by ω_1.

4.2. The even coherent state

In this subsection we shall discuss the even coherent state for the time-dependent frequency converter model given by (1.4). This state is defined for a single mode as follows:

$$|\xi\rangle = N_\alpha \sum_{i=1}^{2} |\alpha e^{i\phi_i}\rangle \qquad \phi_{1,2} = \pi, 2\pi \tag{4.12a}$$

and

$$N_\alpha = \tfrac{1}{2}\exp\left(\tfrac{1}{2}|\alpha|^2\right)\sqrt{\operatorname{sech}|\alpha|^2}\ . \tag{4.12b}$$

Similarly we can define the state for the second mode 'b'. By calculating the quadrature variances for the first mode 'a' we find

$$\overline{\Delta X_1}^2 = \tfrac{1}{4} + \tfrac{1}{2}|\alpha|^2|f(t)|^2[\cos 2(\gamma_+(t) + \gamma(t)) + \tanh|\alpha|^2]$$
$$+\tfrac{1}{2}|\beta|^2|g(t)|^2[\cos 2\gamma_+(t) + \tanh|\beta|^2] \tag{4.13a}$$

$$\overline{\Delta Y_1}^2 = \tfrac{1}{4} - \tfrac{1}{2}|\alpha|^2|f(t)|^2[\cos 2(\gamma_+(t) + \gamma(t)) - \tanh|\alpha|^2]$$
$$-\tfrac{1}{2}|\beta|^2|g(t)|^2[\cos 2\gamma_+(t) - \tanh|\beta|^2] \tag{4.13b}$$

where

$$\gamma(t) = \tan^{-1}(\cos 2\delta \tan I(t)) \tag{4.13c}$$

and

$$f(t) = (\cos I(t) - \mathrm{i}\cos 2\delta \sin I(t))\mathrm{e}^{-\mathrm{i}\gamma_+(t)} \tag{4.14a}$$

$$g(t) = -\mathrm{i}\sin 2\delta \sin I(t)\mathrm{e}^{-\mathrm{i}\gamma_+(t)}\,. \tag{4.14b}$$

From equations (4.13*a*), (4.13*b*) it follows that, as α and $\beta \to 0$, the fluctuation in both quadratures becomes zero, and then the system will reach the minimum uncertainty. However, for large value of α and β the fluctuations in the X_1 and Y_1 quadrature become larger compared with the vacuum-state (or the coherent-state) value which is equal to $\frac{1}{4}$. In the meantime, for $t = 0$ the reduction of the fluctuations occurs in $\overline{\Delta Y_1}^2$, while the fluctuations in $\overline{\Delta X_1}^2$ are enhanced. On the other hand, for $t > 0$ the reduction of the fluctuations starts to appear in the first quadrature $\overline{\Delta X_1}^2$ (see, for example, the case where $\alpha = \beta$), while the fluctuations in the second quadrature $\overline{\Delta Y_1}^2$ is enhanced. This shows that, as a result of the oscillation terms the squeezing will start to exchange between the two quadratures.

5. The correlation function

From the nature of the frequency converter model, the system under consideration is always in coherence, this can be seen if one examines the Glauber second-order correlation function against the coherent state or against the vacuum state. However, the system can show anti-bunching as well as bunching when we measure $g_a^{(2)}(t)$ in the number state. To see this, let us first calculate the $g_a^{(2)}(t)$ which are defined by

$$g_a^{(2)}(t) = 1 + \frac{\overline{\Delta n_a}^2 - \langle n_a\rangle}{\langle n_a\rangle^2}\,. \tag{5.1}$$

From equations (4.1*a*), (4.1*b*) and (5.1) we have

$$g_a^{(2)}(t) = 1 + \frac{2|f(t)|^2|g(t)|^2 n_a(0)n_b(0) - |f(t)|^4 n_a(0) - |g(t)|^4 n_b(0)}{(|f(t)|^2 n_a(0) + |g(t)|^2 n_b(0))^2} \tag{5.2}$$

where $n_a(0)$ and $n_b(0)$ are respectively the photon numbers for modes a and b at the time $t = 0$, while $f(t)$ and $g(t)$ are given by (4.14).

Now if we consider the case when $n_a = n_b = 1$, then equation (5.2) gives

$$g_a^{(2)}(t) = (4\theta^2 \sin^2 I(t) + \sin^2 2I(t))/(1+\theta^2)^2\,. \tag{5.3}$$

It is obvious that the value of the function $g_a^{(2)}(t)$ in the above equation is always $\leqslant 1$, which shows anti-bunching for the system whatever the value of the parameter θ and the oscillating function. On the other hand, if we increase the photon numbers n_a and n_b, in (5.2) then the situation will be different: bunching would appear for some value of both θ and the oscillating function. For example, when $n_a = n_b = 2$, equation (5.2) gives

$$g_a^{(2)}(t) = \tfrac{1}{2} + 3\sin^2 I(t)(\theta^2 + \cos^2 I(t))/(1+\theta^2)^2\,. \tag{5.4}$$

By taking the maximum value of $\sin^2 I(t)$, the system shows partial coherence behaviour, provided $\theta = 1$, where the function $g_a^{(2)}(t) \to 1.25$. The same conclusion is valid, when $\theta = 0$, but for different values of the time. For a fixed value of t, we find that the parameter θ plays the role of decreasing or increasing the value of the function $g_a^{(2)}(t)$. Thus we may conclude from the above analysis that the existence of the parameter θ (which is the result of the integrability condition) does play a role (but not a significant one) when we measure the second-order correlation function with respect to the number state.

We now turn our attention to examining the correlation function $g_a^{(2)}(t)$ by employing the even coherent state. From equations (4.1a) and (4.12) we find

$$\langle n_a(t)\rangle = |f(t)|^2|\alpha|^2 \tanh|\alpha|^2 + |g(t)|^2|\beta|^2 \tanh|\beta|^2 \tag{5.5}$$

$$\begin{aligned}\overline{\Delta n_a}^2(t) - \langle n_a(t)\rangle = {} & |f(t)|^4|\alpha|^4 \operatorname{sech}^2|\alpha|^2 + |g(t)|^4|\beta|^4 \operatorname{sech}^2|\beta|^2 \\ & +2|g(t)|^2|f(t)|^2|\alpha|^2|\beta|^2 \tanh|\beta|^2 \tanh|\alpha|^2 \\ & +\alpha^2\beta^{*2} f^2(t) g^{*2}(t) + \alpha^{*2}\beta^2 g^2(t) f^{*2}(t)\end{aligned} \tag{5.6}$$

when $\alpha = \beta$ the second-order correlation function $g_a^{(2)}(t)$ becomes

$$\begin{aligned}g_a^{(2)}(t) = {} & 1 + 2|f(t)|^2|g(t)|^2 + (f^2(t)g^{*2}(t) + g^2(t)f^{*2}(t)) \coth^2|\alpha|^2 \\ & +(|f(t)|^4 + |g(t)|^4) \operatorname{cosech}^2|\alpha|^2 .\end{aligned} \tag{5.7}$$

For large α equation (5.7) reduces to

$$g_a^{(2)}(t) = 1 + \frac{4\theta^2}{(1+\theta^2)^2} \sin^4 I(t) . \tag{5.8}$$

From equation (5.8) we can conclude that, by invoking the even coherent state to calculate the Glauber second-order correlation function $g_a^{(2)}(t)$, and as a result of the time dependent coupling parameter $\lambda(t)$ and the phase pump $\phi(t)$, the function $g_a^{(2)}(t)$ shows oscillatory behaviour, as well as thermal distribution at the maximum value of the oscillating function when $\theta = 1$. This result cannot be obtained if the parameter θ is zero.

6. Quasi-probability distribution

In this section we pay attention to calculate the quasi-probability phase-space distributions, for one single mode 'a' connected with the correlated squeezed operator (4.3). There are three types of the distribution functions, namely the P-representation, W-Wigner and Q-functions. To find one of these functions we have to calculate the characteristic function $C_p(\xi, t)$ which are defined as follows:

$$C_p(\xi, t) = \mathrm{Tr}(\hat{\rho}(0) \exp(\xi a^\dagger(t)) \exp(-\xi^* a(t))) \tag{6.1}$$

where $\hat{\rho}$ is the density matrix, for the state (4.2). From equations (6.1) and (4.1a) the characteristic function takes the form

$$\begin{aligned}C_p(\xi, t) = {} & \exp[-\sinh^2 r|\xi|^2 + \tfrac{1}{2}(\xi^{*2} f(t)g(t) + \xi^2 f^*(t)g^*(t)) \sinh 2r] \\ & \times \exp(\xi\bar{\alpha}^*(t) - \xi^*\bar{\alpha}(t))\end{aligned} \tag{6.2}$$

where $\bar{\alpha}(t)$ is the mean value of the operator $a(t)$ with respect to the squeezed coherent state (4.2). Having obtained the characteristic function, we are therefore in position to calculate the quasi-probability distribution functions P-representation, W-Wigner and Q-functions. These functions are given by

$$\begin{bmatrix} P(\alpha, t) \\ W(\alpha, t) \\ Q(\alpha, t) \end{bmatrix} = \pi^{-2} \int_{-\infty}^{\infty} \mathrm{d}^2\xi \; \exp(\alpha\xi^* - \alpha^*\xi) \begin{bmatrix} C_P(\xi, t) \\ C_W(\xi, t) \\ C_Q(\xi, t) \end{bmatrix} \tag{6.3}$$

where $C_W(\xi, t) = \exp(-\frac{1}{2}|\xi|^2)\ C_p(\xi, t)$ and $C_Q(\xi, t) = \exp(-|\xi|^2)C_p(\xi, t)$. Note that at $t = 0$ the P-function has the form of the Fourier transform of a Gaussian with negative width. The resulting non-analytic behaviour of the P-representation is a symptom of the non-classical correlations between the a and b modes. On the other hand, for $t > 0$ the P-function can be obtained, but under the restricted condition

$$\operatorname{sech} r < \frac{\theta^2 + \cos 2I(t)}{\theta^2 + 1} \tag{6.4}$$

such that $\theta \in (-\infty, -1)\ U(1, \infty)$. In this case we have

$$P(\alpha, t) = \frac{1}{\pi}(\sinh^4 r - |f(t)|^2|g(t)|^2 \sinh^2 2r)^{-\frac{1}{2}} \exp(-l_1|\bar{\alpha}(t) - \alpha|^2) \tag{6.5a}$$

where

$$l_1 = \frac{[\operatorname{sech}^2 r + |f(t)||g(t)| \operatorname{cosech} 2r \sin(2\gamma_+(t) + \gamma(t) + \nu(t))]}{(\tanh^2 r - 4|f(t)|^2|g(t)|^2)} \tag{6.5b}$$

$\nu(t)$ is an arbitrary time-dependent phase and $\gamma(t)$ is defined by (4.13c). From equation (6.5) we can mention that the P-representation shows Gaussian behaviour with maximum value at $\bar{\alpha}(t) = \alpha$. However, when the value of the parameter θ occurs within the interval $[-1, 1]$, the P-representation does not exist for some values of the time, where the inequality given by (6.4) will be violated. Therefore we can say that the existence of the parameter θ gives us an advantage in finding the P-representation within a certain interval of the parameter θ for all periods of the time.

The other two functions, the W-Wigner and Q-functions are found to be

$$W(\alpha, t) = \frac{2}{\pi}\frac{\operatorname{sech} 2r}{\sqrt{1 - 4\tanh^2 2r|f(t)|^2|g(t)|^2}} \exp(-2l_2|\bar{\alpha}(t) - \alpha|^2) \tag{6.6a}$$

where

$$l_2 = \operatorname{sech} 2r \frac{[1 + 2\tanh 2r|f(t)||g(t)| \sin(2\gamma_+(t) + \gamma(t) + \nu(t))]}{[1 - 4\tanh^2 2r|f(t)|^2|g(t)|^2]} \tag{6.6b}$$

and

$$Q(\alpha, t) = \frac{1}{\pi}(\cosh^4 r - \sinh^2 2r|f(t)|^2|g(t)|^2)^{-\frac{1}{2}} \exp(-2l_3|\bar{\alpha}(t) - \alpha|^2) \tag{6.7a}$$

where

$$l_3 = \operatorname{sech}^2 r\ \frac{[1 + 2\tanh r|f(t)||g(t)| \sin(2\gamma_+(t) + \gamma(t) + \nu(t))]}{[1 - 4\tanh^2 r|f(t)|^2|g(t)|^2]}\,. \tag{6.7b}$$

Equations (6.6) and (6.7) show Gaussian behaviour with maximum value at $\bar{\alpha}(t) = \alpha$. In fact the Gaussian form of these functions (6.7b) means that their contours may be used to map out the phase dependence of the field fluctuations. For example, the contours of the Wigner function give the variances in the field quadratures, while the contours of the Q-function provide the anti-normally ordered variances in the field quadratures.

Now we shall find the mean value of the normally ordered product of an arbitrary number of factors $a^\dagger$ and a. We may express this average in terms of the characteristic function (6.2) by means of the formula

$$\operatorname{Tr}\{\hat{\rho}(t)a^{\dagger n}a^m\} = \left(\frac{\partial}{\partial\xi}\right)^n \left(-\frac{\partial}{\partial\xi^*}\right)^m C_p(\xi, t)|_{\xi=\xi^*=0}\,. \tag{6.8}$$

If we substitute equation (6.2) in (6.8) we find that

$$\mathrm{Tr}(\hat{\rho}(t)a^{\dagger n}a^m) = \left(\frac{1}{2}\sinh 2r\right)^{\frac{1}{2}(m+n)} (f(t)g(t))^{\frac{1}{2}(m-n)} \exp\left[\frac{\mathrm{i}\pi}{2}(m-n)\right]$$
$$\times \sum_{l=0}^{m} \frac{m!n!}{[l!m-l!n-l!]} (\tanh r)^l |f(t)g(t)|^{(n-l)}$$
$$\times H_{(m-l)}\left[\frac{\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi}\bar{\alpha}(t)}{\sqrt{2f(t)g(t)\sinh 2r}}\right] H_{(n-l)}\left[\frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\bar{\alpha}^*(t)}{\sqrt{2f^*(t)g^*(t)\sinh 2r}}\right]. \tag{6.9}$$

As a special case, when $n = m$ we find that

$$\mathrm{Tr}\left[\rho(t)\frac{a^\dagger a!}{(a^\dagger a-n)!}\right] = \left(\frac{1}{2}\sinh 2r\right)^n (n!)^2 \sum_{l=0}^{n} [l!(n-l!)^2]^{-1}$$
$$\times (\tanh r)^l H_l\left(\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\bar{\alpha}(t)}{\sqrt{2f(t)g(t)\sinh 2r}}\right) H_l\left(\frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\bar{\alpha}^*(t)}{\sqrt{2f^*(t)g^*(t)\sinh 2r}}\right). \tag{6.10}$$

Finally we shall calculate the matrix elements of the density operator $\hat{\rho}$ in the n-quantum representation. To do so one may use the W-Wigner function (6.6) together with the formula

$$\langle\tilde{\beta}|\hat{\rho}(t)|\tilde{\beta}\rangle = 2\int \mathrm{d}^2\alpha \ \exp(-2|\tilde{\beta}-\alpha|^2) \ W(\alpha, t). \tag{6.11}$$

However, the matrix elements of the density operator $\hat{\rho}(t)$ can also be deduced from (6.7), since

$$\langle\alpha|\hat{\rho}(t)|\alpha\rangle = \pi Q(\alpha, t). \tag{6.12}$$

By expanding the exponential term in (6.7a) as a power series and using (6.12) we have

$$\langle m|\hat{\rho}(t)|n\rangle = \sqrt{\frac{n!}{m!}}(2l_3(t)\bar{\alpha}(t))^{m-n}(1-2l_3)^n \exp(-2l_3|\bar{\alpha}(t)|^2)$$
$$\times L_n^{(m-n)}\left(-4l_3^2(t)\frac{|\bar{\alpha}(t)|^2}{(1-2l_3(t))}\right) \tag{6.13}$$
$$= \sqrt{\frac{m!}{n!}}(2l_3(t)\bar{\alpha}^*(t))^{n-m}(1-2l_3)^m \exp(-2l_3|\bar{\alpha}(t)|^2)$$
$$\times L_m^{(n-m)}\left(-4l_3^2(t)\frac{|\bar{\alpha}(t)|^2}{(1-2l_3(t))}\right) \tag{6.14}$$

where the $L_m^{(n-m)}(z)$ are the associated Laguerre polynomials, and l_3 is given by (6.7b).

The right-hand side of (6.14) may be obtained from the right-hand side of (6.13) by taking the complex conjugate of the latter and interchanging n and m; this relation is a reflection of the Hermiticity requirement

$$\langle m|\hat{\rho}(t)|n\rangle = (\langle n|\hat{\rho}(t)|m\rangle)^*. \tag{6.15}$$

Now if we set $n = m$, in (6.13) or (6.14) we obtain the probability of finding n quanta in the a mode; thus

$$\langle n|\hat{\rho}(t)|n\rangle = (1-2l_3)^n \exp(-2l_3|\bar{\alpha}(t)|^2) L_n\left[-4\frac{l_3^2(t)|\bar{\alpha}(t)|^2}{(1-2l_3(t))}\right]. \tag{6.16}$$

Note that the matrix elements of the density operator $\hat{\rho}(t)$ can also be rewritten in an alternative form, namely

$$\begin{aligned}\langle m|\hat{\rho}(t)|n\rangle = {} & \frac{1}{\sqrt{l_4(t)}}\exp\left(-\cosh^2 r\frac{|\bar{\alpha}(t)|^2}{l_4(t)}\right)\\ & \times\exp\left[\frac{-\mathrm{i}}{2l_4(t)}|f(t)g(t)|\sinh 2r\left(\bar{a}^{*2}(t)\mathrm{e}^{-\mathrm{i}(2\gamma_+(t)+\gamma(t))}-\bar{\alpha}^2(t)\mathrm{e}^{\mathrm{i}(2\gamma_+(t)+\gamma(t))}\right)\right]\\ & \times\sum_{l=0}^{n}\frac{\sqrt{m!n!}}{[l!m-l!n-l!]}2^{-(l+m)}\left[\frac{(1-4|f(t)||g(t)|)\sinh 2r}{|f(t)g(t)|}\right]^l\\ & \times\exp\left[\frac{\mathrm{i}}{2}(2\gamma_+(t)+\gamma(t))(n-m)\right]\left[\frac{|f(t)||g(t)|\sinh 2r}{(\cosh^4 r-|f(t)g(t)|^2\sinh^2 2r)}\right]^{\frac{1}{2}(m+n)}\\ & \times\exp\left[\mathrm{i}\frac{\pi}{2}(n-m)\right]\\ & \times H_{(n-l)}\left[\frac{\bar{\alpha}^*(t)\cosh^2 r\mathrm{e}^{-\frac{1}{2}\mathrm{i}(2\gamma_+(t)+\gamma(t))}-\mathrm{i}|f(t)||g(t)|\sinh 2r\mathrm{e}^{\frac{1}{2}\mathrm{i}(2\gamma_+(t)+\gamma(t))}}{\mathrm{i}\sqrt{l_4(t)|f(t)||g(t)|\sinh 2r}}\right]\\ & \times H_{(m-l)}\left[\frac{\bar{\alpha}(t)\cosh^2 r\mathrm{e}^{\frac{1}{2}\mathrm{i}(2\gamma_+(t)+\gamma(t))}+\mathrm{i}|f(t)||g(t)|\sinh 2r\mathrm{e}^{-\frac{1}{2}\mathrm{i}(2\gamma_+(t)+\gamma(t))}}{-\mathrm{i}\sqrt{l_4(t)|f(t)||g(t)|\sinh 2r}}\right]\end{aligned} \tag{6.17}$$

where

$$l_4(t) = (\cosh^4 r - \sinh^2 2r|f(t)|^2|g(t)|^2)\,. \tag{6.18}$$

The complication in the above expression is due to the absence of the time-dependent phase $\nu(t)$.

7. Conclusion

In the present paper we have considered the problem of the time-dependent frequency converter with arbitrary coupling parameter and phase pump. The solution of the whole problem is built up on the integrability condition (2.9), which shows continuous periodic energy exchange between the modes with the periods being nonlinear function of time. The main purpose of the present work is to fill the gap of other papers which concentrated on the statistical properties of the system, see for example [11–14]. The paper was divided into two parts, the first part being devoted to giving the exact solution of the wavefunction in the Schrödinger picture. The result was then used to find the wavefunction in the quasi-coherent state. We also managed to deduce the accurate definition for the Dirac operators, which were then used to diagonalize the Hamiltonian (1.4). The Green function for the system was obtained by employing the solution in the Heisenberg picture, and we showed that the results given in [20] can be deduced from the present result. In the second part we examined the squeezing phenomenon, where two different cases were considered. The first case was an examination of the squeezed coherent state, while the second was of the even coherent state. The system showed fluctuations in both cases, as well as exchanging between the quadrature variances. Our consideration was also extended to include the correlation function, which was examined in the above two cases, in which we saw that the parameter

θ plays a dominant role in controlling the system to reach the thermal distribution in the case of the even coherent state. Finally, we considered the quasi-probability distribution functions, where the P-representation was calculated under a restricted condition of the parameter θ. We also showed that the distribution functions exhibit Gaussian behaviour with maximum value at $\bar{\alpha}(t) = \alpha$. The explicit form for the normally ordered product and the matrix elements of the density operator $\hat{\rho}(t)$ were given.

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